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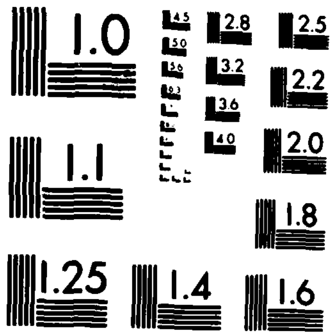

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Ergodicity and Steady-State-Equilibrium Conditions for Markov Chains

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UCT/DEECS/TR-85-1

Department of

Electrical Engineering and Computer Science

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ERGODICITY AND STEADY-STATE EQUILIBRIUM CONDITIONS

FOR MARKOV CHAINS

by

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Abstract

Generalized stationary Markov chains with denumerable state space are considered. For irreducible and aperiodic such chains, some sufficient conditions for ergodicity and steady-state equilibrium are developed. The conditions for ergodicity are generalizations of previously proposed such conditions, and they are more tractable for certain applications.

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1. Introduction

The analysis of several stochastic models gives rise to ergodicity and steady-state equilibrium studies of Markov chains. When a Markov chain is irreducible and aperiodic, with state space, C, and transition probabilities, $\{p_{ji}\}$, a necessary and sufficient condition for ergodicity is that a solution of the following system exists.

$$\pi_j = \sum_{i\varepsilon C} p_{ji}\,\pi_i \;;\; \pi_i \geq 0 \;;\; \forall\, i, \sum_{i\varepsilon C} \pi_i = 1$$

The study of the above system requires explicit knowledge of all the transition probabilities, $\{p_{ji}\}$, and for large dimensionality state spaces, C, the search for its solution becomes practically impossible. Recognizing this fact, several researchers provided simplified ergodicity conditions, for certain classes of Markov chains. Pakes (1969) derived a sufficient condition for ergodicity of Markov chains, $\{X_n\}$, that is solely based on the expected conditional drifts, $E\{X_{n+1}-X_n | X_n=k\}$. Kaplan (1979) provided a criterion for nonergodicity of a Markov chain, which is again based on the expected conditional drifts, and on certain imposed conditions on the transition probabilities. Szpankowski (1981) generalized the conditions given by Pakes and Kaplan, using Lyapunov functions. Szpankowski's approach is especially useful, when the Markov chain state space is not the natural numbers.

In this paper, we generalize Szpankowski's sufficient conditions for ergodicity of Markov chains. In addition, we provide conditions for steady-state equilibrium of irreducible, aperiodic, and ergodic Markov chains, with denumerable state spaces.

2. Ergodicity Conditions

Let $\{X_n\}$ denote a generalized stationary Markov chain, with denumerable state space, C. We allow multidimensional states. Since the state space, C, is denumerable, we assign a unique natural number to each state. Then, the expressions $k\varepsilon C$ and $X_n=k$ denote respectively, the state in C that has been assigned the natural number k, and the nth in time (multidimensional) datum from the chain being identical to the state identified as k. Let, $\{p_{\ell k};\ell,k\varepsilon C\}$, be the set of the stationary transition proba-

bilities of the chain. That is,

$$p_{\ell k} \triangleq P(X_{n+1}=\ell | X_n=k) \quad ; \ \ell,\ k\epsilon C$$

Let us denote,

$$p_{\ell k}^{(n)} \triangleq P(X_n=\ell | X_o=k)$$

Let R denote the real line, and let V: $C \rightarrow R$ be a functional defined on the state space, C, such that,

$$\exists\ \alpha : V(k) \geq \alpha > -\infty \ ; \ \forall\ k\epsilon C \tag{1}$$

Let us then define,

$$S(k) \triangleq E\{V(X_{n+1})-V(X_n) | X_n=k\} = \sum_{\ell \epsilon C} [V(\ell)-V(k)] p_{\ell k} \ ; \ k\epsilon C \tag{2}$$

The quantity S(k) in (2) represents conditional expected drift of the Markov chain, $\{V(X_n)\}$. From now on, whenever the expected value, $E\{f(X_n)\} = \sum_{\ell \epsilon C} f(\ell)P(X_n=\ell)$, is used, for some functional, $f : C \rightarrow R$, the implied assumptions will be that, $\sum_{\ell \epsilon A} f(\ell) = 0 \ ; \ \forall\ A \neq \emptyset$, and that at least one of the partial sums, $\sum_{\ell : f(\ell) \leq 0} f(\ell)P(X_n=\ell)$ and $\sum_{\ell : f(\ell) > 0} f(\ell)P(X_n=\ell)$, is finite. The series, $\sum_{\ell \epsilon C} f(\ell)P(X_n=\ell)$, is then inambiguous, and when it converges, it converges absolutely; that is, $f(X_n)$ is then summable. We now express two propositions.

<u>Proposition 1</u>

Let there exist ℓ in C, and some positive finite number, b, such that,

$$V(\ell) < \infty, \ P(X_o=\ell) = 1 \quad \text{and} \quad S(k) \leq b \ ; \ \forall\ k\epsilon C$$

Then, for all n, the expected values, $E\{|V(X_{n+1})-V(X_n)|\} = \sum_{k\epsilon C} |S(k)| p_{k\ell}^{(n)}$ and

$E\{|V(X_n)|\} = \sum_{k\epsilon C} |V(\ell)| p_{k\ell}^{(n)}$, are both finite. Thus, in conjuction with (1), we then obtain,

$$\alpha \leq E\{V(X_{n+1})\} = E\{V(X_n)\} + E\{V(X_{n+1})-V(X_n)\} = E\{V(X_n)\} + \sum_{k\epsilon C} S(k)p_{k\ell}^{(n)} \qquad (3)$$

Proof

Let, $C^- \triangleq \{k : S(k) \leq 0\}$ and $C^+ \triangleq \{k : S(k) > 0\}$. Then, $0 \leq \sum_{k\epsilon C^+} S(k)p_{k\ell}^{(n)} \leq$ $\leq b \sum_{k\epsilon C} p_{k\ell}^{(n)} = b$; thus, the series $\sum_{k\epsilon C^+} S(k)p_{k\ell}^{(n)}$ is then defined, for all n. That is,

$$\sum_{k\epsilon C^+} S(k)p_{k\ell}^{(n)} < \infty \; ; \; \forall \; n \qquad (4)$$

Let us now turn to the variable $V(X_n)$. Due to (1) and the assumptions in the proposition, we have, $|V(\ell)| < \infty$. Let us select n, and let us temporarily assume that $V(X_n)$ is summable; that is, $\sum_{k\epsilon A} |V(k)|P(X_n=k) < \infty \; ; \; \forall A \subset C$. Then, in conjuction with (1), we obtain,

$$S(k) \geq \alpha - V(k)$$

Thus,

$$0 \geq \sum_{k\epsilon C^-} S(k)p_{k\ell}^{(n)} \geq \alpha \sum_{k\epsilon C^-} p_{k\ell}^{(n)} - \sum_{k\epsilon C^-} V(k)P(X_n=k) > -\infty \qquad (5)$$

So, if $V(X_n)$ is summable, we conclude from (4) and (5),

$$E\{|V(X_{n+1})-V(X_n)|\} = \sum_{k\epsilon C} |S(k)|p_{k\ell}^{(n)} = -\sum_{k\epsilon C^-} S(k)p_{k\ell}^{(n)} + \sum_{k\epsilon C^+} S(k)p_{k\ell}^{(n)} < \infty \qquad (6)$$

Now, since $V(X_{n+1}) = [V(X_{n+1})-V(X_n)] + V(X_n)$, and starting with the initial condition, $|V(\ell)| < \infty$, we can easily complete the proof of the proposition by induction, observing that the summability of $V(X_n)$ and $V(X_{n+1})-V(X_n)$, implies summability of $V(X_{n+1})$.

Remark 1

If in proposition 1, the state ℓ is such that, $0 < P(X_o=\ell) < 1$ and $|V(\ell)| < \infty$, the results in the proposition hold if, $E\{|V(X_n)|\}$, $E\{|V(X_{n+1})-V(X_n)|\}$, $E\{V(X_n)\}$, and $E\{V(X_{n+1})-V(X_n)\}$ are respectively substituted by the conditional expectations, $E\{|V(X_n)|/X_o=\ell\}$, $E\{|V(X_{n+1})-V(X_n)|/X_o=\ell\}$, $E\{V(X_n)/X_o=\ell\}$, and $E\{V(X_{n+1})-V(X_n)/X_o=\ell\}$.

Proposition 2

Let there exist a positive finite number, b, such that, $S(k) \leq b$; $\forall\ k\epsilon C$. Then,

$$\overline{\lim_{n\to\infty}} \sum_{k\epsilon C} S(k)p_{k\ell}^{(n)} \geq 0 \ ; \ \forall\ \ell\epsilon C : P(X_o=\ell) > 0 \tag{7}$$

Proof

Let us assume that (7) is false. Then, there exists some state $\ell\epsilon C$, such that $P(X_o=\ell) > 0$, and for this state there exist $\delta > 0$ and natural number N_δ, such that,

$$\sum_{k\epsilon C} S(k)p_{k\ell}^{(n)} < -\delta \ ; \ \forall\ n \geq N_\delta \tag{8}$$

From expression (3) in proposition 1, modified as in remark 1, in conjuction with (8), we then conclude,

$$E\{V(X_{n+1})|X_o=\ell\} < E\{V(X_n)|X_o=\ell\} - \delta \ ; \ \forall\ n \geq N_\delta$$

And thus,

$$E\{V(X_{N_\delta+k})|X_o=\ell\} < E\{V(X_{N_\delta})|X_o=\ell\} - k\delta \ ; \ \forall\ k \tag{9}$$

But, from proposition 1 we conclude, $E\{V(X_{N_\delta})|X_o=\ell\} < \infty$. Thus, (9) gives then, $\lim_{n\to\infty} E\{V(X_n)|X_o=\ell\} = -\infty$, which is impossible due to (3). We thus conclude that (7) is true.

Remark 2

The statement in proposition 2 holds for any process, $\{X_n\}$, with denumerable state space, whose expected drifts are time invariant, if $E\{|V(X_o)|\} < \infty$, and if

the probability, $p_{k\ell}^{(n)}$, in the proposition, is replaced by the probability, $P(X_n = k)$.

We now express the main result of this section in a lemma. The lemma states a sufficient condition, for ergodicity of a generalized irreducible and aperiodic Markov chain.

Lemma 1

Let $\{X_n\}$ be a generalized irreducible and aperiodic Markov chain, with denumerable state space, C. Let ℓ be some state in C, such that, $P(X_o=\ell) > 0$. Let there exist, $\varepsilon > 0$, $H_1 \subset C$, a set, $\{d_k\}$, of positive finite constants, and a positive and finite constant b, such that,

$$S(k) \leq -\varepsilon; \ \forall \ k\varepsilon H_1 \quad \text{and} \quad -\varepsilon < S(k) \leq b \ ; \ \forall \ k\varepsilon H_2 = C - H_1$$

$$\sum_{k\varepsilon H_2} d_k < \infty \quad \text{and} \quad p_{k\ell}^{(n)} \leq d_k \ ; \ \forall \ n, \ \forall \ k\varepsilon H_2$$

Then, the Markov chain, $\{X_n\}$, is ergodic.

Proof

Since the Markov chain is irreducible and aperiodic, the limit, $\pi_k \triangleq \lim_{n\to\infty} p_{k\ell}^{(n)}$, always exists, and it is independent of the state, ℓ.

Now,

$$\sum_{k\varepsilon C} S(k)p_{k\ell}^{(n)} = \sum_{k\varepsilon H_1} S(k)p_{k\ell}^{(n)} + \sum_{k\varepsilon H_2} S(k)p_{k\ell}^{(n)} \leq -\varepsilon\left[1 - \sum_{k\varepsilon H_2} p_{k\ell}^{(n)}\right] + \sum_{k\varepsilon H_2} S(k)p_{k\ell}^{(n)}$$

$$\to \sum_{k\varepsilon C} S(k)p_{k\ell}^{(n)} \leq -\varepsilon + \sum_{k\varepsilon H_2} [S(k)+\varepsilon]p_{k\ell}^{(n)} \qquad (10)$$

Since $|S(k)+\varepsilon|p_{k\ell}^{(n)} \leq (b+\varepsilon)d_k$; $\forall$ $k\varepsilon H_2$; $\forall n$, and $\sum_{k\varepsilon H_2} d_k < \infty$, the series on the right side of (10) converges uniformly in n. Hence,

$$\overline{\ell im}_{n\to\infty} \sum_{k\varepsilon C} S(k)p_{k\ell}^{(n)} \leq -\varepsilon + \ell im_{n\to\infty} \sum_{k\varepsilon H_2} [S(k)+\varepsilon]p_{k\ell}^{(n)} = -\varepsilon + \sum_{k\varepsilon H_2} [S(k)+\varepsilon]\pi_k \qquad (11)$$

But, for the simultaneous satisfaction of inequality (11), and inequality (7) in proposition 2, it is necessary that, $\sum_{k\varepsilon H_2} [S(k)+\varepsilon]\pi_k > 0$, which implies that $\pi_k > 0$, for some k in H_2. Thus, the chain is ergodic.

The conditions in the lemma are relatively general, and they imply the conditions for ergodicity, given by Szpankowski (1981). In fact, the latter conditions evolve as a corollary of lemma 1, which is expressed below.

Corollary 1

If the set H_2 in the lemma is finite, then $d_k = 1$; $\forall\ k\varepsilon H_2$ satisfies the condition, $\sum_{k\varepsilon H_2} d_k < \infty$. Thus, the irreducible and aperiodic Markov chain, $\{X_n\}$, is then ergodic.

3. Equilibrium Conditions

In this section, we consider steady-state equilibrium conditions, for generalized irreducible and aperiodic Markov chains. We use the same notation and quantities, as in section 2. We first present a proposition.

Proposition 3

Let, $\{X_n\}$, be a generalized, irreducible, and aperiodic Markov chain, with denumerable state space, C. Let there exist a positive and finite constant, b, such that, $S(k) \leq b$; $\forall\ k\varepsilon C$, and let the chain be ergodic. Then,

$$0 \leq \sum_{k\varepsilon C} S(k)\pi_k \leq b \qquad (12)$$

; where,

$$\pi_k \triangleq \ell im_{n\to\infty} p_{k\ell}^{(n)}$$

Proof

Let the subspaces, C^+ and C^-, be defined as in the proof of proposition 1. We then obtain, for given n,

$$\overline{\lim_{n\to\infty}} \sum_{k\in C} S(k)p_{k\ell}^{(n)} = \overline{\lim_{n\to\infty}}\left(\sum_{k\in C^-} S(k)p_{k\ell}^{(n)} + \sum_{k\in C^+} S(k)p_{k\ell}^{(n)}\right) \leq$$

$$\leq \overline{\lim_{n\to\infty}} \sum_{k\in C^-} S(k)p_{k\ell}^{(n)} + \overline{\lim_{n\to\infty}} \sum_{k\in C^+} S(k)p_{k\ell}^{(n)} \tag{13}$$

Since, $0 \leq S(k) \leq b$; $\forall\ k\varepsilon C^+$, and since $\sum_{k\varepsilon C^+} p_{k\ell}^{(n)}$ converges uniformly in n (see Chung (1960), Th. 4), the sum, $\sum_{k\varepsilon C^+} S(k)p_{k\ell}^{(n)}$, converges uniformly in n as well. Thus, $\overline{\lim_{n\to\infty}} \sum_{k\varepsilon C^+} S(k)p_{k\ell}^{(n)} = \sum_{k\varepsilon C^+} S(k)\pi_k$. Since $S(k) \leq 0$; $\forall\ k\varepsilon C^-$, applying Fatou's lemma, we obtain,

$$\overline{\lim_{n\to\infty}} \sum_{k\varepsilon C^-} S(k)p_{k\ell}^{(n)} \leq \sum_{k\varepsilon C^-} S(k)\pi_k \leq 0$$

From the above, in conjuction with (13), we thus obtain,

$$0 \leq \overline{\lim_{n\to\infty}} \sum_{k\varepsilon C} S(k)p_{k\ell}^{(n)} \leq \sum_{k\varepsilon C^-} S(k)\pi_k + \sum_{k\varepsilon C^+} S(k)\pi_k = \sum_{k\varepsilon C} S(k)\pi_k \leq b$$

The proof of the proposition is now complete.

We now express the main result of this section in a lemma.

Lemma 2

Let the generalized Markov chain, $\{X_n\}$, be irreducible, aperiodic, and ergodic, with denumerable state space, C. Let either one of the following conditions hold, where $V(\cdot)$ is a functional as in section 2.

(i) $\sum_{k\epsilon C} |V(k)|\pi_k < \infty$

; where $\pi_k \overset{\Delta}{=} \lim_{n\to\infty} p_{k\ell}^{(n)}$

(ii) There exists positive and finite number, B, such that,

$$E\{|V(X_{n+1})-V(X_n)|/X_n=K\} \leq B \ ; \ \forall \ k\epsilon C$$

Then,

$$\sum_{k\epsilon C} S(k)\pi_k = 0 \qquad (14)$$

Proof

(a) If condition (i) is satisfied, then $V(X_o)$ and $V(X_1)$ are both integrable and they have the same distribution, if $P(X_o = k) = \pi_k$. Thus,

$$0 = E\{V(X_1)-V(X_o)\} = E\{E\{V(X_1)-V(X_o)|X_o\}\} = \sum_{k\epsilon C} S(k)\pi_k$$

which proves the lemma, if condition (i) is true.

(b) Let condition (ii) be satisfied. Via proposition 3, the series $\sum_{k\epsilon C} S(k)\pi_k$ converges absolutely. Thus (Chung (1960), Th. 2),

$$P\left(\lim_{n\to\infty} n^{-1} \sum_{i=0}^{n} S(X_i) = \sum_{k\epsilon C} S(k)\pi_k\right) = 1 \qquad (15)$$

Let us now assume that there exists state ℓ in C, such that, $P(X_o = \ell) = 1$. Let then, T_k ; $k \geq 0$, denote the time of the kth visit to state ℓ. The sequence, $\{T_k\}$, forms a renewal process, and the process, $\{S(X_i)\}$, is regenerative with respect to, $\{T_k\}$. But, due to the ergodicity assumption, we have, $E\{T_1\} < \infty$, and,

$$|S(k)| = |E\{V(X_{n+1})-V(X_n)|X_n=k\}| \leq E\{|V(X_{n+1})-V(X_n)|/X_n=k\} \leq B \ ; \ \forall \ k\epsilon C$$

Therefore,

$$E\left|\sum_{n=0}^{T_1-1} S(X_n)\right| \leq B \cdot E\{T_1\} < \infty \tag{16}$$

From Stidham (1972), and due to (16) we then conclude,

$$P\left(\lim_{n\to\infty} n^{-1}\sum_{i=0}^{n} S(X_i) = E^{-1}\{T_1\}\cdot E\left\{\sum_{n=0}^{T_1-1} S(X_n)\right\}\right) = 1 \tag{17}$$

From (15) and (17), we then conclude,

$$E^{-1}\{T_1\}\cdot E\left\{\sum_{n=0}^{T_1-1} S(X_n)\right\} = \sum_{k\in C} S(k)\pi_k \tag{18}$$

Given, $P(X_o = \ell) = 1$, let us define,

$$Y_n \triangleq \sum_{i=1}^{n} [E\{V(X_i)|X_{i-1}\} - V(X_i)] \; ; \; n \geq 1 \tag{19}$$

$$Z_n \triangleq \sum_{i=0}^{n-1} S(X_i) = \sum_{i=0}^{n-1}[E\{V(X_{i+1})|X_i\} - V(X_i)] =$$

$$= \sum_{i=1}^{n} [E\{V(X_i)|X_{i-1}\} - V(X_{i-1})] =$$

$$= \sum_{i=1}^{n} [(E\{V(X_i)|X_{i-1}\} - V(X_i)) + (V(X_i)-V(X_{i-1}))]$$

$$= \sum_{i=1}^{n} [V(X_i)-V(X_{i-1})] + Y_n = V(X_n)-V(X_o) + Y_n \; ; \; n \geq 1$$

$$\rightarrow Z_n = V(X_n)-V(\ell) + Y_n \; ; \; n \geq 1 \tag{20}$$

Due to proposition 1, we have, $E\{|V(X_i)|\} < \infty; \; \forall \; i$. Due to the Markovian assumption, we have, $E\{V(X_i)|X_{i-1},\ldots,X_o\} = E\{V(X_i)|X_{i-1}\}$. Thus, the process, $\{Y_n\}$, in (19) is a martingale, with respect to the process, $\{X_n\}$, (see Karlin et al (1975),

p. 240, ex. b). We now obtain,

$$|Y_{n+1} - Y_n| = |E\{V(X_{n+1})|X_n\} - V(X_{n+1})| = |E\{V(X_{n+1}) - V(X_n)|X_n\} -$$

$$- [V(X_{n+1}) - V(X_n)]| \rightarrow |Y_{n+1} - Y_n| \leq B + |V(X_{n+1}) - V(X_n)| \rightarrow$$

$$\rightarrow E\{|Y_{n+1} - Y_n|/X_n,\ldots,X_o\} \leq B + E\{|V(X_{n+1}) - V(X_n)|/ X_n,\ldots,X_o\} =$$

$$= B + E\{|V(X_{n+1}) - V(X_n)|/X_n\} \leq 2\ B\ ;\ \forall\ n \geq 1, \text{ due to condition (ii).} \quad (21)$$

Since now T_1 is a Markov time with respect to $\{X_n\}$, and since $E\ \{T_1\} < \infty$, we obtain in conjuction with (21) and corollary 3.1, page 260, in Karlin et al (1975),

$$E\{Y_{T_1}\} = E\{Y_1\} = E\{E\{V(X_1)|X_o\} - V(X_1)\} = 0 \qquad (22)$$

From (20) and (22), we then obtain,

$$E\{Z_{T_1}\} = E\{V(X_{T_1}) - V(\ell) + Y_{T_1}\} = V(\ell) - V(\ell) + E\{Y_{T_1}\} = 0 \qquad (23)$$

From (20) and (23), we conclude, $E\left\{\sum_{i=0}^{T_1-1} S(X_i)\right\} = 0$; expression (18) thus gives,

$$\sum_{k\epsilon C} S(k)\pi_k = 0$$

The proof of the lemma is now complete.

Remark 3

We note that conditions (i) and (ii), in lemma 2, are such that the one does not imply the other. Condition (ii) does not involve limiting probabilities; thus, it may be more applicable in practice. We note that in the proof of lemma 2, the regenerative process, $\{S(X_i)\}$, may be such that, $S(X_i) \geq - D$, for some D positive and finite. Then, the proof works, via the substitution, $S'(X_i) \triangleq S(X_i) + D$. In contrast, the regenerative process, $\{S(X_i)\}$, has been assumed nonnegative in Stidman (1972).

Lemma 2 provides sufficient conditions for the existence of steady-state equilibrium, in generalized, irreducible, aperiodic, and ergodic Markov chains, with denumerable state space.

4. An Example

One of the many applications of Markov chains lies with the analysis of "limited feedback sensing" random access algorithms, for computer-communication data networks. The "limited feedback sensing" class of random access algorithms requires that each user monitor the feedback from the time he generates a new packet, to the time when this packet is successfully transmitted, and it finds numerous applications in many real systems. The algorithms within this class frequently induce irreducible and aperiodic Markov chains, with denumerable state space. Sufficient conditions for the ergodicity of those chains, provide then lower bounds on the throughput of the algorithms. Sufficient conditions for steady state equilibrium, provide the means for the evaluation of useful algorithmic statistical properties. Here, we will use the results in lemma 2 of this paper, to evaluate such properties for one of the algorithms in Vvedenskaya and Tsybakov (1982).

Let us consider algorithm A in Vvedenskaya and Tsybakov. The analysis of the algorithm is facilitated by the concept of a marker, as described in the above reference. The marker can take the integer values, -1, 0, 1, 2, ... If the marker takes the value, -1, at some point in time, it maintains this value, until a collision is encountered. Upon the occurrence of the latter event, the marker takes the value, 1. From that point on, the marker updates its values, following the rules of the algorithm, until it takes again the value, -1, at which point it completes a session. It then repeats the above process. Let time be measured in slot units, and let, M_i, denote the value of the marker at time, i. If $M_i \geq 0$, let Θ_k^i denote the number of packets that at time i are in cell #k of the stack, whose state represents the algorithmic state. If $M_i = -1$, let Θ_{-1}^i denote the number of packets in cell #0 of the stack. The numbers, $\{\Theta_k^i\}$ and $\{M_i\}$, are random variables. Let us

then define the random vector, $X_i \overset{\Delta}{=} [\Theta_o^i, \Theta_1^i, \ldots, \Theta_{M_i}^i, M_i]$. From the operation of the algorithm, it is easily concluded that the process, $\{X_i\}$; $i \geq 0$ is a Markov chain, with state space, $C = \bigcup_{k=-1}^{\infty} G^k$, where,

$$G^k = \begin{cases} \{[\theta_o, \theta_1, \ldots, \theta_k, k] \ ; \ \theta_i \in N_o\}; & k \geq 0 \\ \{[\theta_{-1}, -1] \ ; \ \theta_{-1} \in N_o\}; & k = -1 \end{cases}$$

Since all the sets in G^k above are denumerable, so is C. Furthermore, the process, $\{X_i\}$; $i \geq 0$, is irreducible and aperiodic.

Let, $\{\theta_k\}$ and $\{m_i\}$, denote respectively realizations of the sequences, $\{\theta_k^i\}$ and $\{M_i\}$. Given $k \in C$, let us then define,

$$V(k) \overset{\Delta}{=} \sum_{i=-1}^{m_k} \theta_i \geq 0$$

It can be easily verified that, $E\{|V(X_{i+1}) - V(X_i)|/X_i = k\}$, satisfies condition (ii) in lemma 2, and that,

$$E\{V(X_{i+1}) - V(X_i)/X_i = [\theta_o, \ldots, \theta_m, m]\} = \lambda \ ; \ \text{if } \theta_o \geq 2$$

$$E\{V(X_{i+1}) - V(X_i)/X_i = [\theta_{-1}, -1]\} = \lambda \ ; \ \text{if } \theta_{-1} \geq 2$$

$$E\{V(X_{i+1}) - V(X_i)/X_i = [1, \theta_1, \theta_2, \ldots, \theta_m, m]\} = -1 + \lambda$$

$$E\{V(X_{i+1}) - V(X_i)/X_i = [1, -1]\} = -1 + \lambda$$

$$E\{V(X_{i+1}) - V(X_i)/X_i = [0, \theta_1, \ldots, \theta_m, m]\} = \lambda$$

$$E\{V(X_{i+1}) - V(X_i)/X_i = [0, -1]\} = \lambda$$

; where λ is the intensity of the Poisson user process.

Thus, from lemma 2 and the above, we conclude that in the λ - region, where the

process, $\{X_i\}$; $i \geq 0$, is ergodic, we have,

$$\sum_{k \in C} S(k)\pi_k = \lambda \sum_{k:\theta_o \text{ and } \theta_{-1} \neq 1} \pi_k + (-1+\lambda) \sum_{k:\theta_o \text{ or } \theta_{-1}=1} = 0 \rightarrow$$

$$\rightarrow \sum_{k:\theta_o \text{ or } \theta_{-1} = 1} \pi_k = \lambda$$

The left part of the above equality represents the limiting probability of successful transmission. The equation expresses then the fact that in the λ - region, where the process, $\{X_i\}$; $i \geq 0$, is ergodic, the input traffic rate equals the output traffic rate.

Given k in $\mathcal{C}$, let us now define,

$$V'(k) \triangleq V'([\theta_o,\theta_1,\ldots,\theta_m]) \triangleq m_k \geq -1$$

For the functional, $V'(k)$, above, condition (ii) in lemma 2 is again satisfied, and,

$$E\{V'(X_{i+1}) - V'(X_i)/X_i = [\theta_o,\ldots,\theta_m,m]\} = 1 \quad ; \text{ if } \theta_o \geq 2$$

$$E\{V'(X_{i+1}) - V'(X_i)/X_i = [\theta_{-1},-1]\} = 2 \quad ; \text{ if } \theta_{-1} \geq 2$$

$$E\{V'(X_{i+1}) - V'(X_i)/X_i = [1,\theta_1,\ldots,\theta_m,m]\} = -1$$

$$E\{V'(X_{i+1}) - V'(X_i)/X_i = [0,\theta_1,\ldots,\theta_m,m]\} = -1$$

$$E\{V'(X_{i+1}) - V'(X_i)/X_i = [1,-1]\} = E\{V'(X_{i+1}) - V'(X_i)/X_i = [0,-1]\} = 0$$

Thus, from lemma 2 and the above, we conclude that in the λ - region, where the process, $\{X_i\}$; $i \geq 0$, is ergodic, we have,

$$\sum_{k\epsilon C} S'(k)\pi_k = \sum_{k:\theta_o\geq 2, m\geq 0} \pi_k + 2 \sum_{k:\theta_{-1}\geq 2, m=-1} \pi_k - \sum_{k:\theta_o=1, m\geq 0} \pi_k - \sum_{k:\theta_o=0, m\geq 0} \pi_k = 0 \rightarrow$$

$$\rightarrow \sum_{k:\theta_o=0 \text{ or } \theta_{-1}=0} \pi_k = 2^{-1}\left[1 + \sum_{k:m=-1} \pi_k - 2\lambda\right]$$

The left part of the above equation represents the limiting probability of an empty slot. Also, $\sum_{k:m=-1} \pi_k = E^{-1}\{L\}$, where $E\{L\}$ is the expected session length induced by the algorithm. Thus, the limiting probability of an empty slot can be found from the above equation, as a function of the expected session length.

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